Also, we observe from Eq. (10b) that minimizing Eq. (9) has the effect of minimizing a lower bound on $\kappa_2(H)$. The condition number of the modal matrix $\kappa_2(H)$ is related to an upper bound on the eigenvalue sensitivities, while $\|H\| \cdot \|H^{-1}e(o)\|$ is related to the tightest bound available for the norm of the estimation error as shown by Eq. (6). Hence, this new design approach should yield an observer that exhibits both small eigenvalue sensitivity and good attenuation of the error induced by an initial condition mismatch of known direction. Finally, we remark that minimizing $\kappa_2(H)$ will not, in general, yield good error attenuation. Conversely, minimizing $\|H\| \cdot \|H^{-1}e(o)\|$ will not, in general, yield small eigenvalue sensitivity.

Example

We consider the linearized lateral dynamics of the L-1011 aircraft as described in Ref. 1. The state variables are bank angle, roll rate, yaw rate, and sideslip angle. The inputs are rudder deflection and aileron deflection.

The control law $u = -Kx$ is given by

$$K = \begin{bmatrix} 0 & 0 & -0.689 & 4.56 \\ -13.1 & -3.13 & 0 & 0 \end{bmatrix}$$

In Ref. 1, the authors assumed that only bank angle and roll rate may be measured. Here we assume that yaw rate is also available for measurement. The observer eigenvalues are chosen, as in Ref. 1, to be $\lambda_1 = -3$, $\lambda_2 = -3.5$, $\lambda_3 = -4$, and $\lambda_4 = -4.5$.

To reflect a gust-induced initial sideslip, we chose the initial state vector to be $x(o) = [0, 0, 0, 5]^T$. However, the observer state is initialized to be the zero vector. Thus, an initial condition mismatch exists in sideslip. The observer gain matrix obtained from using the approach of Ref. 1 is truncated to three significant digits and shown in Table 1 along with the eigenvalues. Observe the extreme sensitivity in the eigenvalues that occurs under the extremely small perturbation caused by truncating the observer gains. This implies that the eigenvalues of $(A - LC)$ will also be sensitive to variations in the elements of the matrix A. We compute a new observer gain matrix by minimizing F in Eq. (9). First, the eigenvalues are chosen that determine the subspaces in which each of the eigenvectors must lie. These subspaces are parameterized by a set of 12 scalars that are initialized at their values from the design based upon the approach of Ref. 1. Then the IMSL, Inc., subroutine ZXMIN[3] is used to perform a quasiNewton search with numerically evaluated derivatives. The optimal observer gain matrix is truncated to three significant digits and shown in Table 1 along with the eigenvalues. Observe the insensitivity of the eigenvalues as compared to the previous design.

The time responses of the actual aircraft states are shown in Fig. 1 for three different control laws. These controllers include 1) feedback of the actual states, 2) feedback of estimated states using the observer based upon the approach of Ref. 1, and 3) feedback of estimated states using the new modalized observer. Observe that, for this example, both observers achieve approximately identical estimation error properties. However, the new observer exhibits significantly smaller eigenvalue sensitivity than the previous observer design.

Conclusion

We have proposed an approach for the design of modalized observers that yields an observer with both good error attenuation to an initial condition mismatch of known direction and small eigenvalue sensitivity. This new design method is based on minimizing a cost function that depends on the norm of the observer modal matrix, the condition number of this modal matrix, and directional information about the initial condition mismatch. An example of the lateral dynamics of the L-1011 aircraft is presented, which shows that this new approach yields a better observer design than a previous design method for modalized observers.

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Simple Hybrid Search Technique for Finding Conic Solutions to the Intercept Problem

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Nomenclature

a = semimajor axis
e = eccentricity
f = $1 - [|r_T(\tau + \Delta\tau)|/p](1 - \cos\Delta\nu)$
G = $[|r_T(\tau + \Delta\tau)||r_S(\tau)|/\sqrt{\mu p}]\sin\Delta\nu$
g = gravitational acceleration at the Earth's surface
I_{max} = p-iteration control parameter (limit)
i = inclination
J_{max} = $\Delta\tau$ prediction control parameter (limit)
k = $|r_T(\tau + \Delta\tau)||r_S(\tau)|(1 - \cos\Delta\nu)$
l = $|r_T(\tau + \Delta\tau)| + |r_S(\tau)|$
M = mean anomaly
m = $|r_T(\tau + \Delta\tau)||r_S(\tau)|(1 + \cos\Delta\nu)$
n_X = nonzero, positive integer control parameter for X_{nom}
p = $a(1 - e^2)$
$r_S(\tau)$ = inertial source position vector at time τ
$r_T(\tau)$ = inertial target position vector at time τ
v = $|\dot{r}_S(\tau_L) + \Delta V|$
X_{nom} = $\eta(1/n[\min\{X\}_1 + \max\{X\}_1])$, where $X \epsilon \{\Delta\tau, |\Delta V|\}$
$\{Y\}_N$ = Nth level search set for Y, where $Y \epsilon \{\tau_L, \Delta\tau, |\Delta V|\}$
γ = local horizontal flight-path angle
ΔE = change in eccentric anomaly equivalent to $\Delta\nu$
ΔF = $-i\Delta E(i = \sqrt{-1})$
ΔV = initial velocity impulse applied at source at τ_L
$\Delta\nu$ = $\cos^{-1}\{[r_T(\tau_L + \Delta\tau) \cdot r_S(\tau_L)]/[|r_T(\tau_L + \Delta\tau)||r_S(\tau_L)|]\}$
$\Delta\tau$ = time of flight from launch to intercept
ϵ_X = $\{X\}_1$ membership tolerance
$\eta(x)$ = $x \epsilon \{X\}_1$ if $\{X\}_L = [X]_1$; otherwise $x_j^{\epsilon} \{X\}_1$, where x_j is the nearest neighbor of x
μ = gravitation constant
ν = true anomaly
Σ_N = Nth level solution search space $= \{\tau_L\}_N \times \{\Delta\tau\}_1 \times |\Delta V|\}_1$
τ_L = time of launch
Ω = right ascension of the ascending node
ω = argument of perigee

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Introduction

A TOPIC of classical astrodynamic interest is the intercept problem, i.e., the determination of a conic trajectory between two (source and target) points in space that satisfies time of launch, initial velocity, and time-of-flight constraints. This trajectory is typically found for a known source-target geometry and time of launch by fixing either the initial velocity or time of flight and solving for the unfixed parameter. This paper presents a description of a simple hybrid search (HS) algorithm that can be used to find intercept launch windows and associated trajectories when neither, one, or both of the initial velocity and time-of-flight constraints are tightly bound.

The HS Algorithm

The HS algorithm is a hybrid technique in the sense that a hierarchy of N levels, where here $N = 1, 2, 3$, of heuristics is coupled with an iterative prediction method in order to find an acceptable conic intercept solution within the search space Σ_3. The $N=1$ heuristic is introduced by the user and simply provides (potentially large) bounds on a search space of interest Σ_1 (typically $\{Y\}_1 = [Y]_1$, where $[Y]_1$ is a closed interval over Y).

The $N = 2$ level heuristics are automatically applied to obtain a refined search space $\Sigma_2 \subseteq \Sigma_1$ by using a priori knowledge about the source-target geometry to rapidly isolate a better set $\{\tau_L\}_2 \subseteq \{\tau_L\}_1$ that is likely to provide achievable intercepts within Σ_2. This heuristics level, as well as the meaning of the qualifier "likely," is discussed in detail in the next section.

The working space Σ_3 is finally obtained by using additional application-dependent criteria, e.g., scheduling requirements, resource availability, etc., to construct an ordered preferred set $\{\tau_L\}_3 \subseteq \{\tau_L\}_2$. Because these criteria are application peculiar, they will not be addressed here. If $\{\tau_L\}_3$ is nonempty, then an iterative prediction technique is applied to each $\tau_{L_i} \epsilon \{\tau_L\}_3$ until a solution is found. Should $\{\tau_L\}_3$ be empty, or no solution is found, the search may be reinitiated using a new Σ_1 space.

The prediction technique uses the p-iteration method,[1] along with the source-target geometry associated with τ_{L_i} and an initial time-of-flight guess $\Delta\tau_{\text{nom}}$, in order to obtain an intercept trajectory and the associated ΔV relative to the source. If $|\Delta V|$ is within an $\epsilon_{|\Delta V|}$ neighborhood of some $|\Delta V|_i \epsilon \{|\Delta V|\}_1$, then a Σ_3 solution has been found. Otherwise, the excursion of $|\Delta V|$ from $|\Delta V|_{\text{nom}}$ is used to predict a new time of flight $\Delta\tau_{\text{new}}$ that is consistent with $\{|\Delta V|\}_1$ and satisfies the condition $\Delta\tau_{\text{new}} \epsilon \{\Delta\tau\}_1$. The process is reinitiated using $\Delta\tau_{\text{new}}$ and repeated until a Σ_3 solution is obtained or acceptable iteration control limits are exceeded. The time-of-flight prediction method is developed in a subsequent section.

The HS algorithm is summarized in the following stepwise description:

1) Define source-target geometry and the control parameter set $\{\{n_X\}, \{\epsilon_X\}, I_{\max}, J_{\max}\}$.

2) Invoke $N = 1$ level heuristic to define Σ_1 and compute $\{X_{\text{nom}}\}$.

3) Invoke $N=2$ level heuristic to define Σ_2 (see next section).

4) Invoke $N = 3$ level heuristic to define Σ_3.

5) Select first $\tau_L \epsilon \{\tau_L\}_3$ and initialize J.

6) Set $\Delta\tau = \Delta\tau_{\text{nom}}$.

7) Increase J by one and initialize I.

8) Determine source-target geometry: Compute $r_S(\tau_L)$ and compute $r_T(\tau_L + \Delta\tau)$.

9) Apply the p-iteration method to the source-target configuration of step 8: If convergence to within an $\epsilon_{\Delta\tau}$ neighborhood of $\Delta\tau$ is achieved within $I_{\max}$ iterations, then go to step 10. Otherwise, go to step 12.

10) Compute the ΔV obtained from the p-iteration solution: If $|\eta(|\Delta V|) - |\Delta V|| \leq \epsilon_{|\Delta V|}$ then go to step 13; if $J \geq J_{\max}$, go to step 12.

11) Predict a new time of flight, $\Delta\tau_{\text{new}}$ (see following section): If the conditions $\Delta\tau_{\text{new}} > \max\{\Delta\tau\}_1$ or $\Delta\tau_{\text{new}} < \min\{\Delta\tau\}_1$ have previously occurred for this τ_L, then go to step 12. Set $\Delta\tau = \eta(\Delta\tau_{\text{new}})$ and go to step 7.

12) An acceptable solution for τ_L has not been found: If the Σ_3 search is not complete, go to step 15, otherwise, go to step 14.

13) An acceptable solution has been found.

14) Terminate the search.

15) Select the next $\tau_L \epsilon \{\tau_L\}_3$, initialize J, and go to step 6.

$N = 2$ Heuristics

As mentioned in the previous section, the $N=2$ level heuristics are used to rapidly isolate a better set $\{\tau_L\}_2 \subseteq \{\tau_L\}_1$ that is "likely" to provide achievable intercepts within the Σ_2 search space. The qualifier "likely" is used here to denote that: 1) this isolation process will generally provide a $\{\tau_L\}_2$ containing τ_{L_i} for which there are no Σ_2 solutions, and 2) no τ_{L_i} is contained within the complement of $\{\tau_L\}_2$ in $\{\tau_L\}_1$ with solutions in Σ_2.

This is done by computing for each $\tau_{L_i} \epsilon \{\tau_L\}_1$ the radii of three intercept achievability envelopes $R_j(\tau_{L_i})$. Here $j = 1,2,3$ is the envelope number and

$$R_j(\tau_{L_i}) = \Big\{ |\dot{r}_S(\tau_{L_i})| \cos\alpha + [|\dot{r}_S(\tau_{L_i})|^2(\cos^2\alpha - 1) + (\max\{|\Delta V|\}_1)^2]^{1/2} + [2g(\Delta r_{ij})]^{1/2} \Big\} \Delta\tau_j \tag{1}$$

where

$$\alpha = K \cos^{-1}\left\{ \frac{\dot{r}_S(\tau_{L_i}) \cdot \underset{\sim}{\rho}_j(\tau_{L_i})}{|\dot{r}_S(\tau_{L_i})|\,|\underset{\sim}{\rho}_j(\tau_{L_i})|} \right\}$$

$$\underset{\sim}{\rho}_j(\tau_{L_i}) = r_T(\tau_{L_i} + \Delta\tau_j) - r_S(\tau_{L_i})$$

$$K = \begin{cases} \dfrac{1}{2} & \text{for } \underset{\sim}{\dot{\rho}}_j(\tau_{L_i}) \cdot \dfrac{\underset{\sim}{\rho}_j(\tau_{L_i})}{|\underset{\sim}{\rho}_j(\tau_{L_i})|} \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\Delta r_{ij} = \begin{cases} 0 & \text{for } |r_S(\tau_{L_i})| - |r_T(\tau_{L_i} + \Delta\tau_j)| < 0 \\ |r_S(\tau_{L_i})| - |r_T(\tau_{L_i} + \Delta\tau_j)|, & \text{otherwise} \end{cases}$$

and

$$\Delta\tau_j = \begin{cases} \min\{\Delta\tau\}_1 & \text{for } j = 1 \\ \tfrac{1}{2}[\min\{\Delta\tau\}_1 + \max\{\Delta\tau\}_1] & \text{for } j = 2 \\ \max\{\Delta\tau\}_1 & \text{for } j = 3 \end{cases}$$

If $|\underset{\sim}{\rho}_j(\tau_{L_i})| < R_j(\tau_{L_i})$ for all j, then the intercept is likely to be achievable and $\tau_{L_i} \epsilon \{\tau_L\}_2$.

The expression for R_j has been semiempirically derived using for its form the simple "rate × elapsed time" product relation. The first two terms within the curly braces of Eq. (1) are the positive-sign quadratic solution for the intercept speed at τ_{L_i} in the direction $\underset{\sim}{\rho}_j(\tau_{L_i})$. The use of max $\{|\Delta V|\}_1$ in this solution, along with the third term which reflects additional speed induced by the Earth's gravity, tends to maximize $R_j(\tau_{L_i})$. The K multiplier has been included in the expression for α in order to account for the nonparallel $\Delta V, \underset{\sim}{\rho}_j(\tau_{L_i})$ alignments that are likely to occur when the relative source-target geometries are closing with time. Numerical testing and experience has shown that the use of K tends to minimize the size of $\{\tau_L\}_2$ in accordance with the two conditions above.

Extensive numerical experimentation has been done to verify that within the domains of the experimental Σ_1 search spaces employed, the $N=2$ heuristic generally behaves in a manner consistent with the two conditions associated with the "likely" qualifier. These experiments applied the $N=2$ heuristic to 369,200 space-space, ground-space, and ground-ground source-target configurations to determine if an intercept was likely. The previously described algorithm, excluding steps 3 and 4, was also applied to this set of configurations to find acceptable intercept trajectories. The Σ_1 search-space domains that were employed utilized a variety of $\{\Delta\tau_1\}$ and $\{|\Delta V|\}_1$ search-set combinations that ranged over [0.0, 1000.0] s and [0.5, 50.0] km/s, respectively. In addition, the space configurations used included both orbital and ballistic trajectories with a, e, i, ω, Ω, and M which ranged over [3000, 8000] km, [0.001, 0.90], $[0, \pi]$, $[0, 2\pi]$, $[0, 2\pi]$ and $[0, 2\pi]$, respectively.

Application of the $N = 2$ heuristic indicated that of the 369,200 total configurations examined, 144,008 of them were likely intercept candidates. The results obtained from application of the algorithm, i.e., steps 1 and 5–12 with $\{\tau_L\}_3 = \{\tau_L\}_1$, to the 369,200 configurations showed that no intercepts existed that were not included in the 144,008 found by the $N = 2$ heuristic, and that intercepts were not possible for 16,871 of these.

$\Delta\tau_{new}$ Prediction

The $\Delta\tau_{new}$ prediction technique used in the HS algorithm is based on the first-order expansion

$$\Delta\tau_{new} \cong \Delta\tau + \left[\frac{d(\Delta\tau)}{dp}\right]\delta p \tag{2}$$

The δp change induced in the intercept trajectory by changes in v and γ is given by

$$\delta p = 2p\left(\frac{\delta v}{v} - \tan\gamma\delta\gamma\right) \tag{3}$$

Using the fact that

$$\delta v = \left(\frac{1}{v}\right)\left\{|\Delta V| + \dot{r}_S(\tau)\cdot\frac{\Delta V}{|\Delta V|}\right\}\delta(|\Delta V|) \tag{4}$$

and assuming that $2p\tan\gamma\delta\gamma$ is negligible, Eq. (3) becomes

$$\delta p \cong \left(\frac{2p}{v^2}\right)\left\{|\Delta V| + \dot{r}_S(\tau)\cdot\frac{\Delta V}{|\Delta V|}\right\}\delta(|\Delta V|) \tag{5}$$

Upon substitution of this equation and the well-known expression for $d(\Delta\tau)/dp$, Eq. (2) becomes

$$\Delta\tau_{new} \cong \Delta\tau - \left(\frac{2p}{v^2}\right)\left[\frac{G}{2p} + \frac{3}{2}a(\Delta\tau - G)\frac{k^2 + (2m - 1^2)p^2}{mkp^2} - T\right] \times \left[|\Delta V| + \dot{r}_S(\tau_L)\cdot\frac{\Delta V}{|\Delta V|}\right][|\Delta V|_{nom} - |\Delta V|] \tag{6}$$

where all nonsubscripted quantities on the right correspond to last computed values, $\delta(|\Delta V|)$ has been replaced with $[|\Delta V_{nom}| - |\Delta V|]$, and

$$T = \begin{cases} \sqrt{\dfrac{a^3}{\mu}}\left\{\dfrac{2k\sin\Delta E}{p(k - 1p)}\right\} & \begin{cases}\text{for elliptical inter-}\\ \text{cept trajectories}\end{cases} \\ -\sqrt{\dfrac{(-a)^3}{\mu}}\left\{\dfrac{2k\sinh\Delta F}{p(k - 1p)}\right\} & \begin{cases}\text{for hyperbolic inter-}\\ \text{cept trajectories}\end{cases} \end{cases} \tag{7}$$

Concluding Remarks

A hybrid search algorithm has been developed that couples three levels of heuristics and an iterative prediction technique to isolate conic intercept solutions over potentially large search spaces. This algorithm is generally stable and well-behaved and has recently been used extensively in space-related analysis activities. Sample intercept cases that illustrate the utility and performance of the algorithm, as well as an additional $\Delta\tau_{new}$ prediction method, can be found in Ref. 2.

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